Evaluating the cylindricity of a nominally cylindrical point set

Olivier Devillers, Franco P. Preparata

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Abstract

The minimum zone cylinder of a set of points in three dimensions is the cylindric crown defined by a pair of coaxial cylinders with minimal radial separation (width). In the context of tolerancing metrology, the set of points is nominally cylindrical, i.e., the points are known to lie in close proximity of a known reference cylinder. Using approximations which are valid only in the neighborhood of the reference cylinder, we can get a very good approximation of the minimum zone cylinder. The process provides successive approximations, and each iteration involves the solution of a linear programming problem in six dimensions. The error between the approximation and the optimal solution converges very rapidly (typically in three iterations in practice) down to a limit error of $\frac{8\omega_0^2}{R}$ (where ω_0 is the width and R is the external radius of the zone cylinder).

Keywords: metrology, minimum cylinder, zone cylinder, roundness, cylindricity

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†INRIA, BP93, 06902 Sophia-Antipolis, France.

‡Center for Geometric Computing, Department of Computer Science, Brown University,115 Waterman Street, Providence, RI 02912.

1 Introduction

The aim of metrology is, given an object, to certify that it has the desired shape.

The usual process consists in sampling a set of points $\mathcal{S}$ on the object boundary and to verify that their distance from the surface of a nominal shape is smaller than some tolerance. Of course, one normally deals with the shape and not with the position of the object, so one has to find a position of the reference shape which minimizes that distance.

Evaluating cylindricity is a very important application in metrology. A large fraction of mechanical parts are indeed cylinders, and the precision-mechanical industry is devoting increasing attention to the certification of the cylindricity of objects. It is worth mentioning that, next to planar surfaces, cylindrical surfaces are the most natural products of industrial machining.

International standard (Publication ISO/DIS 12180-1 and -2) specify different kinds of measures designed to evaluate the quality of a set of points. For cylinders (and similarly for circles and spheres), four notions of cylinders are used: the minimum enclosing cylinder, the maximal enclosed cylinder, the minimum zone cylinder, and the least square cylinder. Among these four notions, the zone cylinder is the one that best conforms with the international specifications. The zone cylinder is the cylindrical crown contained between two co-axial cylinders with minimum radial separation (width) and containing all the data points. This notion is analogous to the two-dimensional notion of zone circle. However, whereas resort to the closest- and farthest-point Voronoi diagrams of the data set in two dimension permits an efficient and exact construction of the zone circle, no analogous constructs are known for the zone cylinder. The situation is analogous to the triviality of constructing the circle by three points in the plane and the considerable difficulty of constructing a circular cylinder by five points in space. Due to this computational difficulty, practitioners frequently prefer to adopt the least square cylinder measure because of its relative computational ease.

In all applications, the metrological task operates on a set of measurements of the cylinder surface (the *data points*), and it is reasonable to assume that these points deviate minutely from a nominal cylinder and are more or less uniformly distributed. These assumptions rest on the manufacturing objective to produce a cylindrical object of good quality and on the metrological objective to produce a data set that simplifies the certification task.

In this paper we propose a new approach to the evaluation of cylindricity, which trades computational effort with a negligible loss of accuracy. Specifically our method avoids the analytical difficulty of constructing the zone cylinder of a given set of points, by "linearizing" the problems and obtaining a suitable "object" that contains all the points, and from which a (non-minimal) zone cylinder containing the points can be readily obtained. The resulting zone cylinder, which is obtained by standard linear programming in time proportional to the size of the data point set, yields an excellent approximation of the minimal zone cylinder. We shall bound the error between the width of this zone cylinder and that of the (unknown) zone cylinder of the data points. We shall see that, in general,the quality of the approximation increases with the quality of the "cylindricity" of the physical object (i.e., the quality of its machining process).

Since the construction of the zone circle of a given set of points in the plane (the roundness problem) provides a much simpler two-dimensional setting for our approach to the cylindricity problem and is of interest in its own right (despite the known solution based on Voronoi diagrams), we shall begin by applying our method to the solution of the roundness problem, and shall see that the technique applies equally well to the evaluation of sphericity.

1.1 Roundness

The problem of computing the annulus of minimal area containing a set of points $\mathcal{S}$ in the plane is well known to be reasonably simple, since it is solved by linear programming in the four-dimensional space of the parameters defining an annulus (center, inner and outer radii) in time $O(|\mathcal{S}|) = O(n)$.

The problem of computing the annulus of minimal width ω_0 (and outer radius R_0) containing $\mathcal{S}$ is more difficult and has been extensively studied. Rivlin [6] showed that the minimum-width annulus is defined by 4 points, 2 on the inner circle and 2 on the outer circle and that inner/outer points alternate around the center. In other words, the center of the annulus is at the intersection of an edge of the nearest-neighbor Voronoi diagram and an edge of the farthest-neighbor Voronoi diagram, so that the superposition of these two diagrams give in general an $O(n^2)$ algorithm. This bound was improved to $O(n^{\frac{3}{2}+\varepsilon})$ by Agarwal and Sharir [3]. Agarwal *et al.* also proposed an approximation algorithm which gives an annulus of width smaller than $(1+\epsilon)$ times the optimal width in time $O(n \log n + \frac{n}{\epsilon^2})$ [1]. Garcia, Ramos and Snoeyink[4] and Ramos [5] address the special but very important case where the set $\mathcal{S}$ of points is "almost round": if the circular order of the points around the center is known, then in some relevant domain there is only one local optimum of the width of the annulus (in term of the center) and this optimum can be computed in time $O(n \log n)$. Under the restriction that the points belong to an annulus whose width is one tenth of its radius, the problem is almost LP-type [7] and can be solved in randomized $O(n)$ time.

In Section 2, we define as *2D-dense sample* a set of points obtained by sampling a circle so that no angular sector of width $\pi/2$ is empty, and prove the following result:

Theorem 2 *Given a 2D-dense sample $\mathcal{S}$ of a circular object of nominal radius R, a zone circle (minimum-area annulus) of $\mathcal{S}$ can be computed in time $O(|\mathcal{S}|)$, such that the difference between its width and the width ω_0 of the minimum zone circle of $\mathcal{S}$ is smaller than $3\frac{\omega_0^2}{R_0}$.*

This shows that the method is very adequate in most metrology applications, and that its validity increases with the quality of the mechanical object (smaller ω_0).

1.2 Cylindricity

The natural generalization of the roundness problem in three dimensions is the computation of the minimum-width spherical shell, since this width expresses the sphericity of the set of points. Unfortunately, cylinders are much more complex to manipulate than spheres, and problems such that minimum enclosing cylinder, maximum enclosed cylinder or minimum zone cylinder are very difficult. Agarwal *et al.* proposed a solution of the minimum enclosing cylinder problem [2] in $O(n^{3+\epsilon})$ time, involving the lower envelope of algebraic surfaces in 4D and parametric search.

In Section 3 we fully develop our method and propose an approximate scheme exhibiting excellent behavior if the points are realistically assumed to lie in close proximity of some known vertical cylinder and the sampling satisfies some reasonable assumption of denseness (to be defined as a *3D-dense sample*). Our approach uses two devices. First, instead of searching cylinders, we search another family of quadrics (one-sheet elliptic hyperboloids with circular horizontal sections) which are good approximations of cylinders when the axis is nearly vertical. Second, as for the roundness problem, we solve the analog of a "minimum area" problem, by conveniently "linearizing" the problem. This device affords an efficient linear-time construction of an approximating hyperboloid, from which a valid zone cylinder is immediately obtained. We also show that the process can be iterated with rapid convergence and we prove:

Theorem 6 *Given a 3D-dense sample $\mathcal{S}$ of a cylindrical object and a zone cylinder of radius R and width smaller than $R/4$, a zone cylinder of $\mathcal{S}$ can be computed by iterated linear programming such that the difference between its width and the width ω_C of the minimum zone cylinder of $\mathcal{S}$ is smaller than* $\frac{5.7\omega_C^2}{R}$.

2 The roundness problem

We begin with a discussion of the much simpler two-dimensional roundness problem. Our conclusions will provide the setting for the cylindricity problem.

As is well known, the circularity of an object is judged on the basis of the size of the annulus enclosing all the measured points, and there are two alternative definitions of annulus size: The first is the *area*, the second is the *radial separation* or *width* (in the latter case, the annulus is referred to in metrology as the zone circle). As is intuitive, we shall show that in the reasonable hypothesis that the measured points deviate minutely from an ideal circle, the two definition of size give rise to minutely different solutions.

In a suitable system of coordinates, the minimum-width annulus, of outer radius R and width ω_0, is defined by the inequalities:

$$(R - \omega_0)^2 \leq x^2 + y^2 \leq (R)^2$$

Its area satisfies the inequality

$$\sigma_0 = \pi \left(R^2 - (R - \omega_0)^2 \right) \leq 2\pi R \omega_0$$

Next we consider the minimum-area annulus. In the same frame of reference, each point of the measured set must satisfy the two constraints

$$x^2 + y^2 - 2\alpha x - 2\beta y \leq R^2 + 2R\eta = (R + \eta')^2 - \alpha^2 - \beta^2$$

$$x^2 + y^2 - 2\alpha x - 2\beta y \geq R^2 + 2R\tau = (R + \tau')^2 - \alpha^2 - \beta^2$$

Since the area of the just defined annulus is

$$\sigma = 2\pi R(\eta - \tau)$$

we recognize (as is well known) that the minimum-area annulus is obtained by minimizing the (linear) objective function $(\eta-\tau)$ subject to linear constraints on the parameters, a linear-programming problem.

Denoting by ω the width of the minimum-area annulus, by definition we have $\sigma \leq \sigma_0$ and $\omega_0 \leq \omega$. We now wish to establish that, under reasonable hypotheses, ω is an excellent approximation to the minimum ω_0.

We know that the minimum-width annulus is determined by four points of $\mathcal{S}$, two on the inner and two on the outer circle. Therefore if the two annuli are concentric, they necessarily coincide. It follows that any difference between the two is due to a displacement of their respective centers. Specifically, let (α, β) denote the center of the minimum-area annulus, so that the distance between the centers is $d = \sqrt{\alpha^2 + \beta^2}$.

Expanding $\sqrt{R^2+\xi}$ to third-order terms,we obtain

$$\sqrt{R^2+\xi} \simeq R + \frac{\xi}{2R} - \frac{\xi^2}{8R^3} + \frac{5\xi^3}{128R^5}.$$

Form this, it is easy to establish that for $\xi \in [-R^2, R^2]$:

$$R + \frac{\xi}{2R} - \frac{\xi^2}{8R^3} + \frac{5\xi^3}{16R^5} \leq \sqrt{R^2+\xi} \leq R + \frac{\xi}{2R}$$

Thus we have

$$\begin{aligned}
\omega &= \sqrt{R^2 + 2R\eta + \alpha^2 + \beta^2} - \sqrt{R^2 + 2R\tau + \alpha^2 + \beta^2} \\
&\leq R + \eta - R - \tau + \frac{(2R\tau + d^2)^2}{8R^3} + \frac{5(2R\tau + d^2)^3}{16R^5} \\
&= \frac{\sigma}{2\pi R} + \frac{\tau^2}{2R} + \frac{\tau d^2}{2R^2} - \frac{5\tau^3}{2R^2} + \frac{d^4}{8R^3} - \frac{15\tau d^4}{8R^4}
\end{aligned}$$

Our next objective is to bound from above the error term $\frac{\tau^2}{2R} + \ldots$ in terms of ω_0. To this end, we introduce some reasonable assumption on the distribution of the measured points, as expressed by the following definition:

Definition 1 *A set of points obtained by sampling a circular object is a* 2D-dense sample *if there is no empty sector with respect to the center of angular width* $\frac{\pi}{2}$.

With this assumption, and referring to Figure 1, we obtain the inequalities:

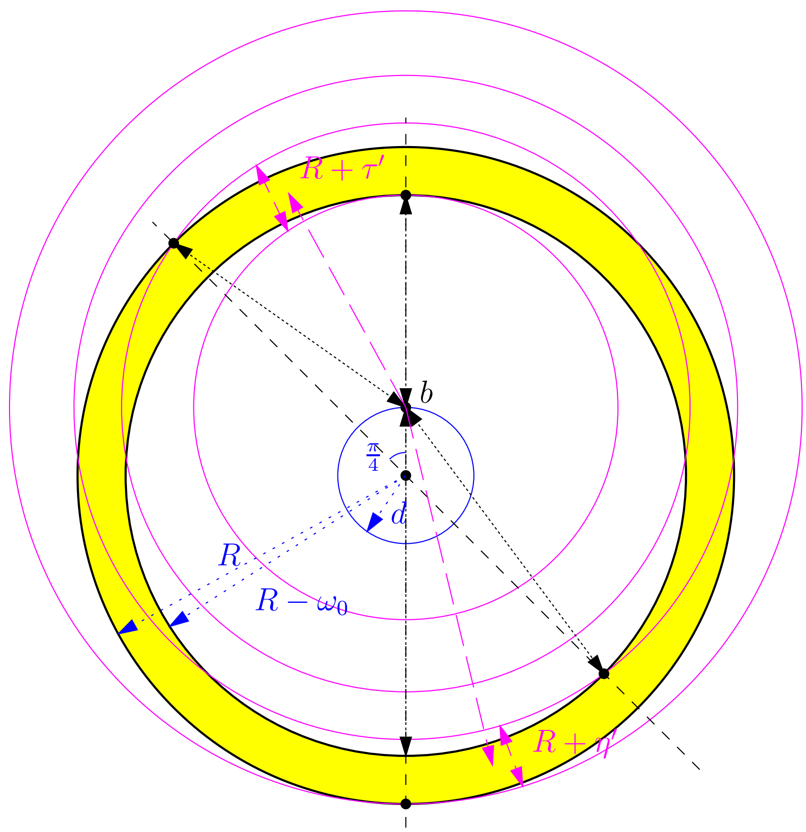

Figure 1: Minimum width annulus (shaded,centered at O) and possible positions of the minimum area annulus (centered at b).

$$
\begin{aligned}
R-\omega_0-d \;\; &\le R+\tau' \;\; \le \sqrt{\left(\frac{d}{\sqrt{2}}\right)^2+\left(R-\frac{d}{\sqrt{2}}\right)^2} \\
&\qquad\qquad\;\; = \sqrt{d^2+R^2-\sqrt{2}Rd} \\
&\qquad\qquad\;\; \le R \\
R+d \;\; &\ge R+\eta' \;\; \ge \sqrt{\left(\frac{d}{\sqrt{2}}\right)^2+\left(R-\omega_0+\frac{d}{\sqrt{2}}\right)^2}
\end{aligned}
$$

$$= \sqrt{d^2 + R^2 + \sqrt{2}Rd + \omega_0^2 - 2\omega_0\left(R + \frac{d}{\sqrt{2}}\right)}$$

From these we can bound σ from below, i.e.:

$$\begin{aligned}
\sigma &= \pi(R+\eta')^2 - \pi(R+\tau')^2 \\
&\geq \pi(2\sqrt{2}Rd + \omega_0^2 - 2\omega_0 R - \sqrt{2}\omega_0 d) \\
&\geq \pi\sqrt{2}(2R - \omega_0)d - 2\pi\omega_0 R
\end{aligned}$$

Using the fact that $2\pi R\omega_0 \geq \ \sigma_0 \geq \sigma$, we obtain

$$\begin{aligned}
d &\leq \frac{4R\omega_0}{\sqrt{2}(2R-\omega_0)} \\
&\leq \frac{2\omega_0(2R - \omega_0 + \omega_0)}{\sqrt{2}(2R-\omega_0)} \\
&\leq \sqrt{2}\omega_0 + \frac{\omega_0^2}{\sqrt{2}R}
\end{aligned}$$

Finally, observing that $0 > \tau \geq \tau' \geq -\omega_0 - d$ we conclude that $|\tau| \leq \omega_0 + d$, so that

$$\frac{\tau^2}{2R} \leq \frac{(\omega_0+d)^2}{2R} \leq \frac{(3+2\sqrt{2})\omega_0^2}{2R} + O\left(\frac{\omega_0^3}{R^2}\right)$$

$$\frac{\tau d^2}{2R^2} - \frac{5\tau^3}{2R^2} + \frac{d^4}{8R^3} - \frac{15\tau d^4}{8R^4} \leq O\left(\frac{\omega_0^3}{R^2}\right)$$

Therefore the error term can be bounded and, since $(3+2\sqrt{2})/2 < 3$, we get:

$$\omega_0 \leq \omega \leq \omega_0\left(1 + 3\frac{\omega_0}{R} + O\left(\frac{\omega_0^2}{R^2}\right)\right).$$

We summarize the preceding discussion as follows:

Theorem 2 *Given a 2D-dense sample $\mathcal{S}$ of a circular object of nominal radius R, a zone circle (minimum-area annulus) of $\mathcal{S}$ can be computed in time $O(|\mathcal{S}|)$, such that the difference between its width and the width ω_0 of the minimum zone circle of $\mathcal{S}$ is smaller than $3\frac{\omega_0^2}{R_0}$.*

This result validates our prior claim on the quality of the approximation.

To provide a quantitative appreciation of this fact in metrological applications, if $R = 1$cm and $\omega = 10\mu$m, the resulting error on the width is at most 30nm.

Higher dimension

The dimension is not explicitly used in all computations above, thus we can deduce the following result. Given a set $\mathcal{S}$ of points in d dimensions, if the minimal-width spherical shell containing $\mathcal{S}$ has width ω_0 and there is a point of $\mathcal{S}$ in any cone of angle $\frac{\pi}{2}$ with vertex at the center, then the spherical shell minimizing the difference of the square radii of the inner and outer spheres has width ω such that

$$\omega_0 \leq \omega \leq \omega_0 \left(1 + 3\frac{\omega_0}{R}\right)$$

This spherical shell can be computed by linear programming in $d+2$ dimensions.

3 The cylindricity problem

We assume that the nominal cylinder has equation

$$x^2 + y^2 = R^2, \quad -h \leq z \leq h$$

A *zone cylinder* (domain limited by two coaxial cylinders of identical height $2h$) is defined by the mid-point $(\alpha, \beta, 0)$ on the common axis of the two cylinders, by the direction (u, v, h) of the axis, by the external radius $R+\eta$, and by the internal radius $R+\tau$. Thus the six parameters (all having dimension of a length) $u, v, \alpha, \beta, \eta, \tau$ define a point $\mathbf{p} = (u, v, \alpha, \beta, \eta, \tau)$ of a 6-dimensional space $\mathcal{P}$, to which a unique zone cylinder $Cyl(\mathbf{p})$ corresponds. The origin of $\mathcal{P}$ corresponds to the nominal cylinder (obviously with radial separation 0). Since the six parameters are expected to be very small, we shall restrict our attention to a neighborhood of the origin.

3.1 Equations of a circular cylinder and of its associated hyperboloid

Consider the axis of parameters (u, v, α, β). The distance $\delta(x, y, z)$ (or δ for short) of a point (x, y, z) of space from this axis is the length of the difference between the vector $w = (x - \alpha, y - \beta, z)$ and its projection on the axis of length $\frac{w \cdot a}{||a||}$ (where $a = (u, v, h)$). Thus we have

$$||w||^2 = \left(\frac{w \cdot a}{||a||}\right)^2 + \delta^2$$

and

$$\begin{aligned}
\delta^2 &= ||w||^2 - \left(\frac{w \cdot a}{||a||}\right)^2 \\
&= \frac{1}{h^2 + u^2 + v^2}\Big[(h^2 + u^2 + v^2)\Big((x-\alpha)^2 + (y-\beta)^2 + z^2\Big) \\
&\qquad -((x-\alpha)u + (y-\beta)v + hz)^2\Big] \\
&= \frac{1}{h^2 + u^2 + v^2}\Big[(h^2 + v^2)x^2 + (h^2 + u^2)y^2 - 2uvxy \\
&\qquad +(u^2 + v^2)z^2 - 2huxz - 2hvyz + 2h(u\alpha + v\beta)z \\
&\qquad -2(h^2\alpha + v^2\alpha - uv\beta)x - 2(h^2\beta - uv\alpha + u^2\beta)y \\
&\qquad +h^2(\alpha^2 + \beta^2) + v^2\alpha^2 + u^2\beta^2 - 2uv\alpha\beta\Big]
\end{aligned} \tag{1}$$

The zone cylinder $Cyl(u, v, \alpha, \beta, \eta, \tau)$, illustrated in Figure 2, is defined by the inequalities:

$$\begin{aligned}
(R + \tau)^2 \leq \quad & \tfrac{1}{h^2+u^2+v^2}\left[(h^2 + v^2)x^2 + (h^2 + u^2)y^2 - 2uvxy\right. \\
& +(u^2 + v^2)z^2 - 2huxz - 2hvyz + 2h(u\alpha + v\beta)z \\
& -2(h^2\alpha + v^2\alpha - uv\beta)x - 2(h^2\beta - uv\alpha + u^2\beta)y \\
& \left.+h^2(\alpha^2 + \beta^2) + v^2\alpha^2 + u^2\beta^2 - 2uv\alpha\beta\right] \qquad \leq (R + \eta)^2
\end{aligned} \tag{2}$$

The equation $\delta^2 = R^2$ describes a cylinder of radius R and axis of parameters (u, v, α, β). Using this equation to determine the parameters of a cylinder through five points or the zone cylinder through six points gives rise

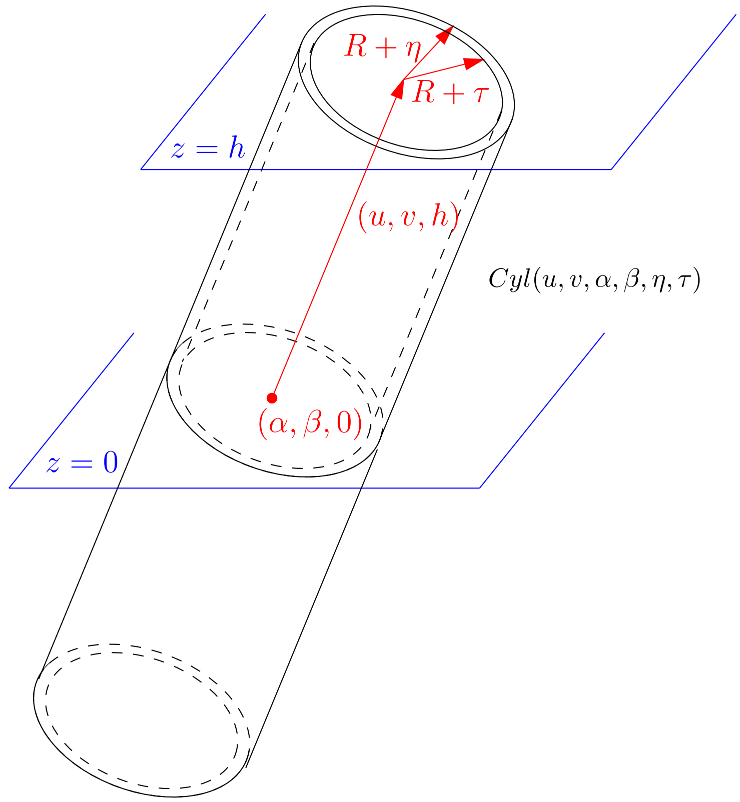

Figure 2: Parameters of a cylinder

to a system of degree-4 equations[1]. This kind of systems need sophisticated algebraic geometry tools for their solution. It is therefore appropriate to resort to suitable approximations.

We assume that $\mathbf{p} = (u, v, \alpha, \beta, \eta, \tau)$ is in a neighborhood of the origin of size $O(\theta)$ we will define precisely in Section 3.3. Rewriting Equation (1) in a form where only first-order terms in θ are explicit, we obtain:

$$\delta^2 = x^2 + y^2 - 2\frac{u}{h}xz - 2\frac{v}{h}yz - 2\alpha x - 2\beta y + O(\theta^2) \tag{3}$$

Given the cylinder $\delta^2 = R^2$, the quadric surface

$$R^2 = x^2 + y^2 - 2\frac{u}{h}xz - 2\frac{v}{h}yz - 2\alpha x - 2\beta y$$

[1]This parameterization is not the most convenient one for determining the equation of a cylinder through given points and it is used here for the purpose of our approximation.

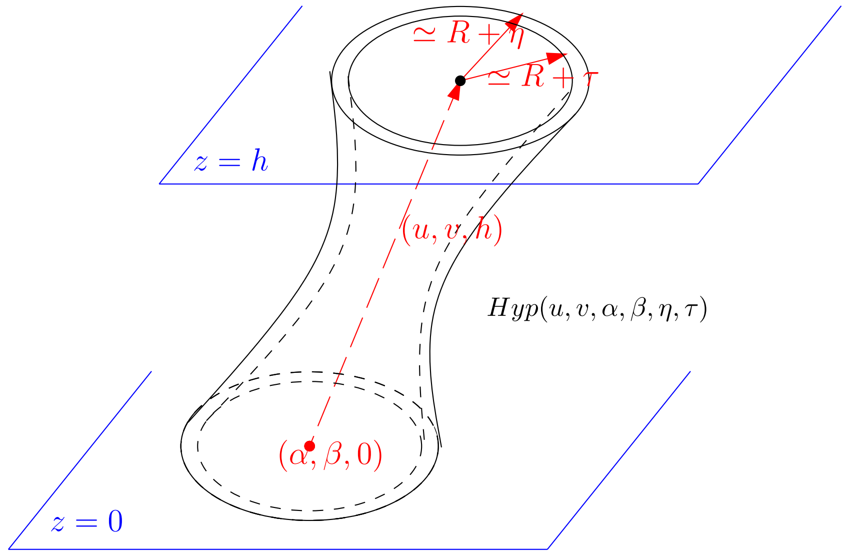

Figure 3: Cylinder approximation by a hyperboloid

is a one-sheet elliptic hyperboloid, referred to as the hyperboloid *associated* to the cylinder. Moreover, given $\mathbf{p} \in \mathcal{P}$, consider the geometric object $Hyp(\mathbf{p})$ defined by

$$R^2 + 2R\tau \leq \quad x^2 + y^2 - 2uxz/h - 2vyz/h - 2\alpha x - 2\beta y \quad \leq R^2 + 2R\eta \tag{4}$$

This region of space is delimited by two coaxial one-sheet elliptic hyperboloids, and its horizontal sections are annuli. It will be referred to as a *zone hyperboloid* (see Figure 3).

Following Equation (3) we compute a second order approximation of δ^2:

$$\begin{aligned}&\delta^2 - (x^2 + y^2 - 2ux - 2vyz - 2\alpha x - 2\beta y) \\ &= \quad (ux - vy)^2/h^2 + (u^2 + v^2)(z^2)/h^2 + 2(u\alpha + v\beta)z/h + \alpha^2 + \beta^2 + O(\theta^3)/h\end{aligned} \tag{5}$$

Clearly, a cylinder and its associated hyperboloid are two distinct geometric objects with a common axis. If this axis (i.e., its defining parameters (u, v, α, β)) is known, then performing a transformation of coordinates that brings the z-axis to coincide with the common axis, both objects have equations $x^2 + y^2 = R^2$, that is, they coincide. This observation is the key to our approach. In fact, rather than searching over the set of zone cylinders, we perform the much easier tasks of searching over the set of zone hyperboloids as defined above. If, as it happens in general, the axis of the minimal zone cylinder has nonzero parameters (u, v, α, β), then it does not coincide with the axis of the minimum zone hyperboloid (as computed for the given data set), nor do the two objects coincide. However, if we can prove that these two axes are reasonably close, then we can bring the z-axis to coincide with the (known) axis of the minimum zone hyperboloid, and be assured that the (unknown) axis of the minimum zone cylinder is brought closer to the z-axis than it was initially. Iterating this process, we can approach the cylinder axis with very good precision (for small width),

3.2 Computation of the minimal zone hyperboloid

As previously stated, our approximation approach consists of searching, rather than circular cylinders, hyperboloids with circular horizontal sections. Specifically, we are looking for a zone hyperboloid, parameterized by six parameters as described by Equation (4) containing all the points of $\mathcal{S}$, i.e., for a point $p_i \in \mathcal{S}$ the parameters of the hyperboloid must satisfies the two inequalities:

$$-\frac{2x_i z_i}{h}u - \frac{2y_i z_i}{h}v - 2x_i\alpha - 2y_i\beta + x_i^2 + y_i^2 \leq R^2 + 2R\eta \simeq (R+\eta)^2$$

$$-\frac{2x_i z_i}{h}u - \frac{2y_i z_i}{h}v - 2x_i\alpha - 2y_i\beta + x_i^2 + y_i^2 \geq R^2 + 2R\tau \simeq (R+\tau)^2$$

This formulation yields $2n$ constraints in a six-dimensional space, which can be solved in time $O(n)$ by a linear program minimizing the objective function $\eta - \tau$.

Incidentally, the same approach enables the determination of the minimum enclosing hyperboloid or of the maximum inscribed hyperboloid (by linear programming in 5 dimensions).

3.3 Error analysis: Distance between $Cyl(\mathbf{p})$ and $Hyp(\mathbf{p})$

Let (x, y, z) be a point on the outer hyperboloid of $Hyp(\mathbf{p})$, i.e. (x, y, z) satisfies the equation:

$$-\frac{2xz}{h}u - \frac{2yz}{h}v - 2x\alpha - 2y\beta + x^2 + y^2 = R^2 + 2R\eta.$$

The distance $\delta(x, y, z)$ of this point from the axis, satisfies $\delta^2 = R^2 + 2R\eta + O(\theta^2)$ by Equation (3) and Inequality (4). Using now the explicit expression of the error $O(\theta^2)$ given by Equation (5) we have:

$$\begin{aligned}\delta^2 \;\; &= \;\; R^2 + 2R\eta + (ux - vy)^2/h^2 + (u^2 + v^2)z^2/h^2 \\ &\quad\; +2(u\alpha + v\beta)z/h + \alpha^2 + \beta^2 + O(\theta^3)/h\end{aligned}$$

We now characterize the neighborhood of the origin of $\mathcal{P}$ to be used in our approximation. We assume that:

- the distance between the axis of our object and the origin in the plane $z = 0$ is at most θ,
- the horizontal projection of vector (u, v, h) defining the axis direction has length smaller than θ,
- η and τ are also of the same order of magnitude θ.

Next we make some reasonable assumptions on the quality of the physical object and of the sampling process:

- for $-h \leq z \leq h$, we have $||(x, y)|| \leq R + O(\theta)$ and $\theta \leq R \leq h$.

Definition 3 *A set of points obtained by sampling a cylindrical object is a* 3D-dense sample *if the at height $z \in \{-h, 0, h\}$ it contains a 2D-dense sample.*

We formalize these hypotheses as follows:

Hypotheses:

$$
\begin{array}{lrcl}
(i) & u^2+v^2, \alpha^2+\beta^2 & \leq & \theta^2 \\
(ii) & |\eta|, |\tau| & \leq & \theta \\
(iii) & ||(x,y)|| & \leq & R+O(\theta) \\
(iv) & \theta & \leq & R \leq h \\
 & \multicolumn{3}{l}{\forall z_0 \in \{-h,0,h\} \text{ and } \forall \text{ sector } W} \\
(v) & \multicolumn{3}{l}{\text{in plane } z=z_0 \text{ of angular width}} \\
 & \multicolumn{3}{l}{\geq \frac{\pi}{2};\ W \cap \mathcal{S} \neq \emptyset}
\end{array} \tag{6}
$$

The domain defined by (6i,ii,iii) will be denoted $\mathcal{N}(\theta)$. From these conditions we derive the following bounds:

$$
\begin{array}{rlcll}
0 \leq & |ux-vy| & \leq & \theta(R+O(\theta)) & \text{from } (6i, iii) \\
0 \leq & (ux-vy)^2 & \leq & \theta^2(R^2+O(R\theta)) \leq \theta^2 h^2 + O(h\theta^3) & \\
0 \leq & |u\alpha+v\beta| & \leq & \theta^2 & \text{from } (6i)
\end{array}
$$

We recall that $R+\frac{\xi}{2R}-\frac{\xi^2}{8R^3}+\frac{5\xi^3}{16R^5} \leq \sqrt{R^2+\xi} \leq R+\frac{\xi}{2R}$for $\xi \in [-R^2, R^2]$. Therefore using above values of δ^2 and Hypotheses (6):

$$
\begin{aligned}
\delta^2 &\leq R^2+2R\eta+5\theta^2+\frac{1}{h}O(\theta^3) \\
\delta &\leq R+\frac{1}{2R}\left(2R\eta+5\theta^2+\frac{1}{h}O(\theta^3)\right) \\
&\leq R+\eta+\frac{5\theta^2}{2R}+\frac{1}{h^2}O(\theta^3)
\end{aligned} \tag{7}
$$

and

$$
\begin{aligned}
\delta^2 &\geq R^2+2R\eta-2\theta^2+\frac{1}{h}O(\theta^3) \\
\delta &\geq R+\frac{1}{2R}\left(2R\eta-2\theta^2+\frac{1}{h}O(\theta^3)\right)-\frac{1}{8R^3}\left(2R\eta+O(\theta^2)\right)^2-\frac{5R^3O(\theta^3)}{16R^5} \\
&\geq R+\eta-\frac{\theta^2}{R}-\frac{4R^2}{8R^3}\eta^2+\frac{1}{Rh}O(\theta^3) \qquad \text{from } (6ii) \\
&\geq R+\eta-\frac{3\theta^2}{2R}+\frac{1}{h^2}O(\theta^3)
\end{aligned} \tag{8}
$$

Thus, the cylinder of axis (u,v,α,β) and whose constant term in the equation is $R^2+2R(\eta+\frac{5\theta^2}{2R})$ encloses the outer hyperboloid of $Hyp(u,v,\alpha,\beta,\eta,\theta)$

and the cylinder with constant term $R^2 + 2R(\eta - \frac{3\theta^2}{2R})$ is inscribed in this hyperboloid. Similarly, a cylinder of axis (u, v, α, β) and with constant term $R^2+2R(\tau+\frac{5\theta^2}{2R})$ encloses the inner hyperboloid and the cylinder with constant term $R^2 + 2R(\tau - \frac{3\theta^2}{2R})$ is inscribed in it.

This is summarized as follows:

Theorem 4 *Under the hypothesis $\mathbf{p} \in \mathcal{N}(\theta)$, the zone cylinder $Cyl(\mathbf{p} + (0, 0, 0, 0, \frac{5\theta^2}{2R}, -\frac{3\theta^2}{2R}))$ encloses the zone hyperboloid $Hyp(\mathbf{p})$.*

3.4 Relationship between optimal zone hyperboloid and optimal zone cylinder

Consider in $\mathcal{P}$ the sets

$$\begin{aligned} \mathcal{H} = \quad & \{\mathbf{p} \in \mathcal{P} : \mathcal{S} \subset Hyp(\mathbf{p})\} \\ \text{and} \quad \mathcal{C} = \quad & \{\mathbf{p} \in \mathcal{P} : \mathcal{S} \subset Cyl(\mathbf{p})\}. \end{aligned}$$

Since $\mathcal{H}$ is defined by a set of linear constraints, it is a convex polytope in $\mathcal{P}$.

Let $Hyp(\mathbf{p}_H)$ be the minimal zone hyperboloid (computed by the linear program as explained in Section 3.2), that is, $\mathbf{p}_H$ is the extremum of the convex set $\mathcal{H}$ in the direction $\mathbf{v} = (0, 0, 0, 0, 1, -1)$ (since the linear program's objective function is $\eta - \tau$). Let $Cyl(\mathbf{p}_C)$ be the minimum zone cylinder, that is, the global extremum of the set $\mathcal{C}$. Our present objective is to obtain a neighborhood of $\mathbf{p}_H$ in $\mathcal{P}$ that is guaranteed to contain $\mathbf{p}_C$.

Let $\mathbf{p}_e = (0, 0, 0, 0, \frac{5\theta^2}{2R}, -\frac{3\theta^2}{2R})$, $\mathbf{p'}_e = (0, 0, 0, 0, -\frac{3\theta^2}{2R}, \frac{5\theta^2}{2R})$ and $\mathbf{p}$ a point on the boundary of $\mathcal{H}$. From Equations (7) and (8) we conclude that $\mathbf{p} + \mathbf{p}_e$ is inside $\mathcal{C}$ and $\mathbf{p} + \mathbf{p'}_e$ is outside $\mathcal{C}$. In fact $Cyl(\mathbf{p} + \mathbf{p}_e)$ encloses $Hyp(\mathbf{p})$ and therefore $\mathcal{S}$; on the other hand, there exists a point $q \in \mathcal{S}$ which belongs to the boundary of $Hyp(\mathbf{p})$ (otherwise $Hyp(\mathbf{p})$ could not be minimal), so that q is outside $Cyl(\mathbf{p} + \mathbf{p'}_e)$ which consequently cannot contain $\mathcal{S}$ (see Figure 4). This statement holds under Hypotheses (6). The conclusion is that, in the relevant neighborhood $\mathcal{N}(\theta)$ of the origin, the boundary of $\mathcal{C}$ is sandwiched between two translations of the boundary of $\mathcal{H}$ by vectors $\mathbf{p'}_e$ and $\mathbf{p}_e$ (see Figure 5).

We also know that in ordinary space $Cyl(\mathbf{p}_C)$, by definition of optimal solution, has smaller width than $Cyl(\mathbf{p}_H + \mathbf{p}_e)$. In $\mathcal{P}$ this fact is interpreted as follows. Since $\omega = \eta - \tau$ is the width, each width value is associated with a hyper-plane in $\mathcal{P}$ orthogonal to the vector $\mathbf{v}$. Therefore, $\mathbf{p}_C$ lies in the half-space $\mathcal{Z}$ delimited by the hyper-plane passing by $\mathbf{p}_H + \mathbf{p}_e$ with inner normal

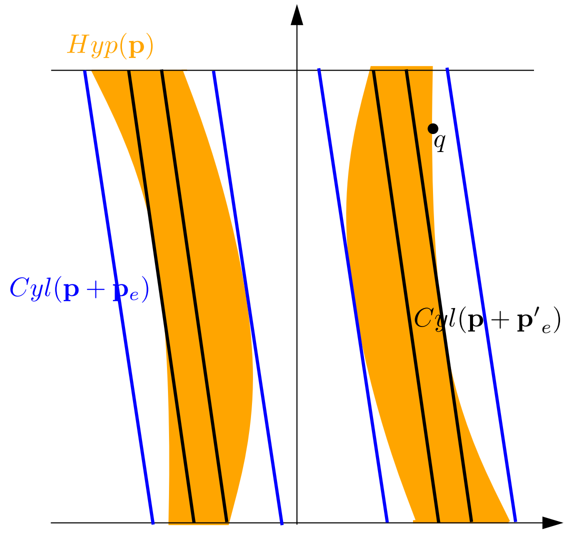

Figure 4: Positions of cylinders and hyperboloids

$\mathbf{v}$; the equation of this hyperplane is $\eta - \tau = \omega_H + \frac{8\theta^2}{2R} = \omega_Z$. Let $\mathcal{H} + \mathbf{p}'_e$ denote the domain obtained by translating $\mathcal{H}$ by $\mathbf{p}'_e$. Then $\mathcal{Z} \cap (\mathcal{H} + \mathbf{p}'_e)$ defines a small domain $\mathcal{K}$ that is guaranteed to contain $\mathbf{p}_C$ (see, again, Figure 5). $\mathcal{K}$ is clearly above the plane $\eta - \tau = \omega_H - \frac{4\theta^2}{R}$. It follows that the difference between the respective widths ω_C and ω_H of $Cyl(\mathbf{p}_C)$ and of $Hyp(\mathbf{p}_H)$ is at most $\frac{8\theta^2}{2R}$:

$$|\omega_H - \omega_C| = |(\eta_H - \tau_H) - (\eta_C - \tau_C)| \leq \frac{4\theta^2}{R} \tag{9}$$

We now wish to (over)estimate the size of $\mathcal{K}$. To this end we establish necessary conditions for $\mathbf{p}$ to be in $\mathcal{K}$. Assuming that $\mathbf{p} \in \mathcal{K}$, as in Section 2 (see Figure 1) in appropriate sections of $Hyp(\mathbf{p})$ we can bound the radii of the annulus in terms of the corresponding radii of $Hyp(\mathbf{p}_H)$. Let d be the horizontal distance between the axis of $Hyp(\mathbf{p})$ and the axis of $Hyp(\mathbf{p}_H)$ at some height $z_0 \in \{-h, 0, h\}$. Note that $\sqrt{R^2 + 2R\tau + \rho^2}$, and $\sqrt{R^2 + 2R\eta + \rho^2}$ are the radii of the annulus for $Hyp(\mathbf{p})$ at $z = z_0$, where $\rho^2 = (\alpha + uz_0/h)^2 + (\beta + vz_0/h)^2$. We similarly define $\rho_H^2 = (\alpha_H + u_H z_0/h)^2 + (\beta_H + v_H z_0/h)^2$. Using the hypothesis ($6v$) of sufficiently dense sampling at $z = z_0$ we obtain:

$$\left(\sqrt{R^2 + 2R\eta_H + \rho_H^2} + d\right)^2 \quad \geq R^2 + 2R\eta + \rho^2 \quad \geq \left(\frac{d}{\sqrt{2}}\right)^2 + \left(\sqrt{R^2 + 2R\tau_H + \rho_H^2} + \frac{d}{\sqrt{2}}\right)^2$$

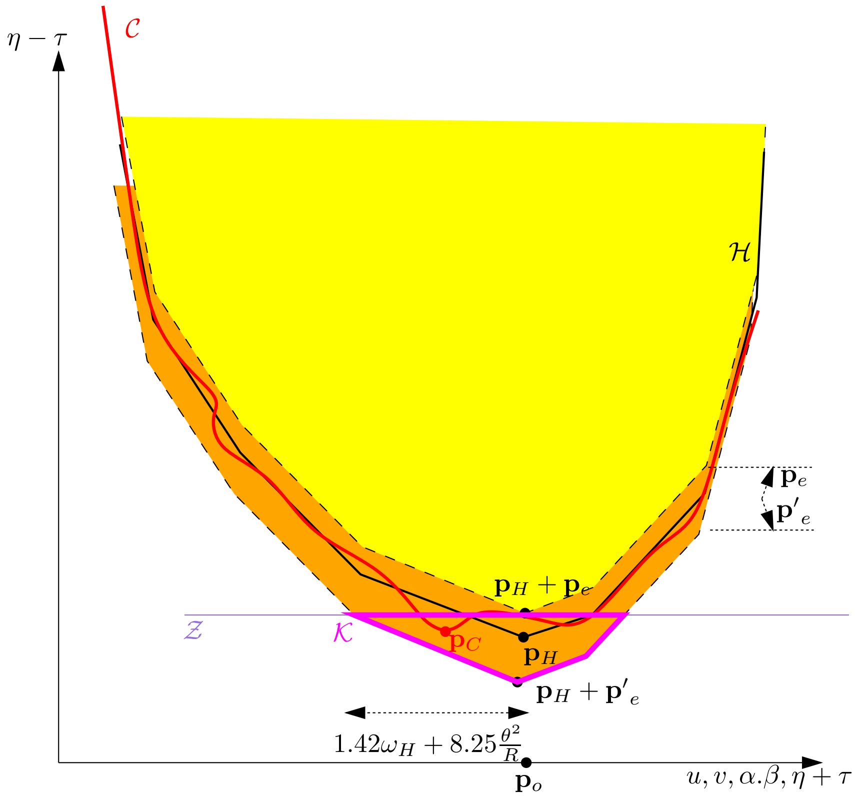

Figure 5: How $\mathbf{p}_C$ is approximated by $\mathbf{p}_H$

$$\left(\sqrt{R^2 + 2R\tau_H + \rho_H^2} - d\right)^2 \quad \leq R^2 + 2R\tau + \rho^2 \quad \leq \left(\frac{d}{\sqrt{2}}\right)^2 + \left(\sqrt{R^2 + 2R\eta_H + \rho_H^2} - \frac{d}{\sqrt{2}}\right)^2$$

From these we derive the following bounds on η and τ:

$$\begin{aligned}
\eta &\geq \tau_H + \frac{d^2 + \rho_H^2 - \rho^2}{2R} + \frac{d\sqrt{R^2 + 2R\tau_H + \rho_H^2}}{\sqrt{2}R} \\
&\geq \tau_H + \frac{d^2 + \rho_H^2 - \rho^2}{2R} + \frac{d}{\sqrt{2}} + \frac{d\tau_H}{\sqrt{2}R} + O\left(\frac{\theta^3}{R^2}\right) \qquad (10) \\
\tau &\leq \eta_H + \frac{d^2 + \rho_H^2 - \rho^2}{2R} - \frac{d\sqrt{R^2 + 2R\eta_H + \rho_H^2}}{\sqrt{2}R}
\end{aligned}$$

$$
\begin{aligned}
&\leq \ \eta_H + \frac{d^2+\rho_H^2-\rho^2}{2R} - \frac{d}{\sqrt{2}} - \frac{d\eta_H}{\sqrt{2}R} + O\left(\frac{\theta^3}{R^2}\right) && (11)\\
\eta &\leq \ \eta_H + \frac{d^2+\rho_H^2-\rho^2}{2R} + \frac{d\sqrt{R^2+2R\eta_H+\rho_H^2}}{R} &&\\
&\leq \ \eta_H + d + \frac{d^2+\rho_H^2-\rho^2+2d\eta_H}{2R} + O\left(\frac{\theta^3}{R^2}\right) && (12)\\
\tau &\geq \ \tau_H + \frac{d^2+\rho_H^2-\rho^2}{2R} - \frac{d\sqrt{R^2+2R\tau_H+\rho_H^2}}{R} &&\\
&\leq \ \tau_H + d + \frac{d^2+\rho_H^2-\rho^2-2d\tau_H}{2R} + O\left(\frac{\theta^3}{R^2}\right) && (13)
\end{aligned}
$$

Now, since $\mathbf{p} \in \mathcal{K}$, we have $\eta - \tau \leq \omega_Z$, i.e.,:

$$\omega_Z \geq \eta - \tau \ \ \geq \ \ \tau_H - \eta_H + \sqrt{2}d + \frac{\tau_H+\eta_H}{\sqrt{2}R}d + O\left(\frac{\theta^3}{R^2}\right)$$

Using $(6ii)$,

$$\omega_Z = \omega_H + \frac{4\theta^2}{R} \ \ \geq \ \ -\omega_H + \sqrt{2}d - \frac{\sqrt{2}\theta}{R}d + O\left(\frac{\theta^3}{R^2}\right)$$

we derive:

$$
\begin{aligned}
d\sqrt{2}\left(1-\frac{\theta}{R}\right) &\leq \ 2\omega_H + \frac{4\theta^2}{R} &&\\
d\sqrt{2} &\leq \ \left(2\omega_H + \frac{4\theta^2}{R}\right)\left(1+\frac{\theta}{R}+O\left(\frac{\theta^2}{R^2}\right)\right) &&\\
d &\leq \ \sqrt{2}\omega_H + \frac{8\theta^2}{\sqrt{2}R} + O(\theta^3)/R && (14)
\end{aligned}
$$

To apply this result to $z_0 = -h, 0, h$, we note that for these values of z the distances between the axes of $Hyp(\mathbf{p}_H)$ and $Hyp(\mathbf{p})$ are, respectively, the lengths of the vectors:
$(\alpha - u - \alpha_H + u_H, \beta - v - \beta_H + v_H)$,
$(\alpha - \alpha_H, \beta - \beta_H)$ and
$(\alpha + u - \alpha_H - u_H, \beta + v - \beta_H - v_H)$
Each of these distances satisfies bound (14), so that:

$$\sqrt{(u-u_H)^2+(v-v_H)^2}, \sqrt{(\alpha-\alpha_H)^2+(\beta-\beta_H)^2} \leq \sqrt{2}\omega_H + \frac{5\theta^2}{\sqrt{2}R} \quad (15)$$

These inequalities bound the size of $\mathcal{K}$ in the first four dimensions of $\mathcal{P}$. It remains to bound it in dimensions η and τ. We use inequalities (10)-(13). Specifically, from (12) and (13) we respectively obtain

$$\begin{aligned}
\eta - \eta_H \leq \quad & d + \tfrac{d^2+\rho_H^2-\rho^2+2d\tau_h}{2R} & \leq d + \frac{9\theta^2}{2R} \\
-\tau + \tau_H \leq \quad & d - \tfrac{d^2+\rho_H^2-\rho^2-2d\tau_h}{2R} & \leq d + \frac{5\theta^2}{2R}
\end{aligned}$$

which, using (14), are combined as

$$\begin{aligned}
-\tau + \tau_H, \eta - \eta_H \quad & \leq \quad d + \frac{9\theta^2}{2R} \\
& \leq \quad \sqrt{2}\omega_H + \frac{(5\sqrt{2}+9)\theta^2}{2R}
\end{aligned}$$

Similarly, subtracting η_H from both sides of (10), τ_H from both sides of (11), and using $\omega_H = \tau_H - \eta_H$, we obtain

$$\begin{aligned}
\eta - \eta_H \quad & \geq \quad -\omega_H + \frac{d}{\sqrt{2}} + \frac{d^2+\rho_H^2-\rho^2+\sqrt{2}d\tau_H}{2R} \\
-\tau + \tau_H \quad & \geq \quad -\omega_H + \frac{d}{\sqrt{2}} - \frac{d^2+\rho_H^2-\rho^2-\sqrt{2}d\tau_H}{2R}
\end{aligned}$$

Using (14), we combine these inequalities as

$$-\tau + \tau_H, \eta - \eta_H \quad \geq \quad -\omega_H - \frac{(1+2\sqrt{2})\theta^2}{R}$$

We have therefore obtained the following bounds of $\mathcal{K}$ in the η and τ dimensions:

$$|\eta - \eta_H|, |\tau - \tau_H| \quad \leq \quad \sqrt{2}\omega_H + \frac{(5\sqrt{2}+9)\theta^2}{2R} \quad (16)$$

Theorem 5 *Given a 3D-dense sample $\mathcal{S}$ of a cylindrical object and a zone cylinder of radius R and width 2θ, a zone cylinder of $\mathcal{S}$ can be computed in time $O(|\mathcal{S}|)$, such that the difference between its width and the width ω_C of the minimum zone cylinder of $\mathcal{S}$ is smaller than $\frac{4\theta^2}{R}$. This zone cylinder approaches the minimal one with an error smaller than $1.414\omega_C + \frac{14\theta^2}{R}$ on the internal and external radii and on the position of the axis.*

3.5 Iterative approximation

The preceding arguments establish that, given a nominal cylinder of radius R and assuming that the minimum zone cylinder, enclosing the measured set $\mathcal{S}$, is in a small neighborhood of the nominal cylinder, we can compute a zone cylinder enclosing $\mathcal{S}$. In our terminology, the latter is $Cyl(\mathbf{p}_H + \mathbf{p}_e)$, whereas $Cyl(\mathbf{p}_C)$ is the minimum zone cylinder and $Cyl(\mathbf{0})$ is the (zero-width) nominal cylinder. The above hypothesis is formalized by saying that $\mathbf{p}_C \in \mathcal{P}$ belongs to a neighborhood of the origin defined by $u^2+v^2, \alpha^2+\beta^2 \leq \theta^2$ and $|\eta|, |\tau| \leq \theta$, for sufficiently small θ.

The quality of the approximation depends of the distance between $\mathbf{0}$ and $\mathbf{p}_C$. Ideally, if the first four coordinates of $\mathbf{p}_C$ are 0, then $\mathbf{p}_H = \mathbf{p}_C$, and the problem is solved exactly by $Hyp(\mathbf{p}_H)$. Therefore our objective is to move the origin of $\mathcal{P}$ closer to $\mathbf{p}_C$ by an iterative process. Since only zero-width nominal cylinders are allowed, the new origin must belongs to the plane $\eta = \tau$.

Let $\theta_1 = \theta$. If we choose as a new origin $\mathbf{p}_1 = (u_H, v_H, \alpha_H, \beta_H, \frac{\eta_H+\tau_H}{2}, \frac{\eta_H+\tau_H}{2})$ (so that $Cyl(\mathbf{p}_1)$ is the new nominal cylinder), Equations (15) and (16) shows that in the new frame of reference the parameter vector $\mathbf{p'}_C$ of the minimum zone cylinder satisfies the bounds:

$$\begin{aligned} \sqrt{u_C'^2 + v_C'^2}, \sqrt{\alpha_C'^2 + \beta_C'^2} &\leq \sqrt{2}\omega_H + \frac{5\theta^2}{\sqrt{2}R} \\ |\eta'|, |\tau'| &\leq \sqrt{2}\omega_H + \frac{(5\sqrt{2}+9)\theta^2}{2R} \end{aligned}$$

This implies that $\theta_2 = \sqrt{2}\omega_H + \frac{(5\sqrt{2}+9)\theta^2}{2R}$ can now be used as the parameter bounding the neighborhood of the origin; if $\theta_2 < \theta_1$, then the process can be iterated in the new parameterization corresponding to the nominal cylinder $C(\mathbf{p}_1)$.

Assuming ω small enough, the i-th iteration will give a solution with an error of $\left(\frac{(5\sqrt{2}+9)\theta}{2R}\right)^{2^i-2}\frac{4\theta}{R}\theta$ on the width. In other words, if $\frac{(5\sqrt{2}+9)\theta}{2R} \leq 1$, i.e., $2\theta \leq R/4$ this process converges to $\mathbf{p}_C$ very rapidly.

The assumption that ω is small enough, upon which the stated convergence is contingent, is equivalent to saying that in the definition of θ_{i+1} the term in θ_i is large with respect to the one in ω_H. When the accuracy improves θ_i decreases and converges to $\sqrt{2}\omega_C$. The upper bound on the error converges to $\frac{4\sqrt{2}\omega_C^2}{R}$. This is summarized as follows:

Theorem 6 *Given a 3D-dense sample $\mathcal{S}$ of a cylindrical object and a zone cylinder of radius R and width smaller than $R/4$, a zone cylinder of $\mathcal{S}$ can be computed by iterated linear programming such that the difference between its width and the width ω_C of the minimum zone cylinder of $\mathcal{S}$ is smaller than $\frac{5.7\omega_C^2}{R}$.*

4 Metrology application

Measuring cylindricity in metrology involves an apparatus consisting of a turntable, a probe, and the support of the probe, as illustrated in Figure 6.

Such a system does not directly measure three-dimensional coordinates. Rather, the probe measures variations of the radius as the table turns or as the probe support slides vertically. This measurement involves three different axes: the axis of rotation of the table, the axis of the cylindrical object and the axis of the probe support. Ideally, all these axes are parallel, but in practice they are not. Conventionally, the axis of the turntable is taken as the z-axis; the deviation of the axis of the probe support is corrected by calibration, and the axis of the cylinder is the real unknown we are looking for.

This system gives three outputs: the height of the probe z, the angle of the table γ and the probe value ρ; these parameters are transformed into the three-dimensional point $((R+\rho)\cos\gamma, (R+\rho)\sin\gamma, z)$. Since the probe measures variations of the distance from the axis of the turntable, getting three-dimensional points involves the knowledge of some nominal radius R.

In a first order approximation, this nominal radius does affect the width of the set of points. We can remark that in the iterative process, described in previous section, we can take advantage of the knowledge of successive

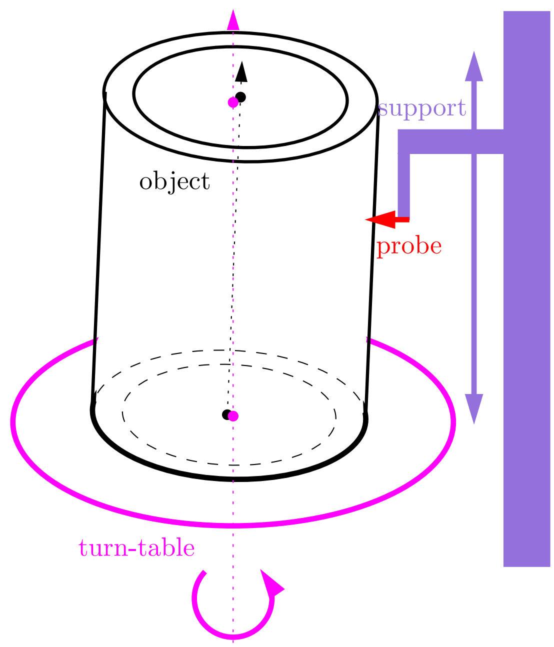

Figure 6: A physical metrology system

nominal cylinders to improve the accuracy of the transformation of the triples (ρ, γ, z) to three dimensional points.

We shall now consider some typical values occurring in metrology systems and compute the resulting error on the width of the minimal zone cylinder. Our parameters are the cylinder height and radius, (of the order of a few centimers), the offset from the cylinder axis (of the order of a small fraction of a millimeter), the machining spread (or "width", of the order of a few μm).The precision of the probe can be as small as 5 nm. In a table below we shall illustrate two examples: The first is a pessimistic case (large positioning error and machining spread), and the error is reduced to about 2% in 4 iterations. The second example is more realistic, and in 3 iterations the error is reduced below the precision of the sensor.

h (mm)	R (mm)	position error	ω (μm)	iteration	θ (μm)	final error
100	5	100μm	10	1	100	8 μm
				2	30	0.7 μm
				3	15.6	0.2 μm
				4	14.5	0.17 μm
				∞	14.2	0.16μm
100	10	50μm	1	1	50	1μm
				2	3.4	4.7nm
				3	1.4	0.8nm
				∞	1.4	0.8nm

5 Experimental results

6 Experimental results

We report below results obtained with our technique, with various numbers of iterations, on two qualitatively different artifacts, and compare them with analogous results obtained by other method currently used for cylindricity evaluation in metrological applications, such as the least square (LS) method, where cylindricity is measured by the deviations of the data points from the least-square cylinder, or the minimum-zone least-square cylinder (LS-MCZ), where the cylindricity results from a minimum-zone optimization starting from a least-square axis. It must be pointed out that the quoted methods only provide upper bounds on the cylindricity, while our method yields a precise evaluation with a bound on the error, i.e. with a certificate of quality. **J'ai donc toujours un problme ici: pour le deuxime objet la 3eme itration me dit que la cylindricit est 19.03 a 0.01 prs et la 4eme que c'est 19.47 a 0.01 prs. Ce n'est pas compatible!**

With reference to our technique, for each iteration we report:
— the evaluated cylindricity,
— the upper bound on the error on the cylindricity obtained by Theorem 5,
— the upper bound θ on the parameters on the cylinder, where this bound is provided at first iteration, and computed using Theorem 5 thereafter,
— the quantity $\max(\sqrt{\alpha^2+\beta^2}, \sqrt{u^2+v^2}, |\eta|, |\tau|)$, denoted $|\mathbf{p}|$ and used to compute θ at the next iteration. This technique has been tried on real data points obtained by actual measurement. For the first object, of very coarse

quality, the six points determining the minimum zone hyperboloid remain identical through all iterations of our method, as was to be expected. The iterations basically improve the curvature of the hyperboloid, so that the end result closely approximates a cylinder. The second object, on the other hand, is of very good quality and is much more interesting. The high quality suggests that the entire set of data points are nearly co-cylindrical, so that the six points defining the minimum zone hyperboloid change from iteration to iteration.

The behavior of the cylindricity values through the iterations is, at first sight, somewhat puzzling, since we intuitively expect some sort of "covergence", and therefore a monotonically decreasing sequence of cylindricity values. Upon reflection, however, we completely justify the observed behavior and fully understand the features of the method. As established in the preceding sections, our method evaluates a minimum zone hyperboloid (i.e., a hyperboloid with minimum radial separation) of a very well defined type, namely, with annular *horizontal* sections. In other words, the resulting hyperboloid is determined (by the data points and) by the accidental position of the z-axis. Only when the z-axis coincides with the resulting hyperboloid axis does the hyperboloid coincide with a minimum-zone cylinder, After modifying the frame of reference, a different set of six points defines the new minimum-zone hyperboloid, and there is no guarantee that the resulting cylindricity be smaller. However, and this is the strength of the method, through successive iterations the error exhibits monotonic convergence and enables us to obtain a sequence of values with an associated guarantes. In our example, Iteration 5 yields the best result with a cylindricity 19.01 ± 0.01, very close to the result 19.03 ± 0.01 of Iteration 3 and superior to the result 19.47 ± 0.01 of Iteration 6.

—————-

We propose below the results obtained with our technique with different number of iterations and we compare it with results obtained by least square methods currently used for cylindricity evaluation **??????? Explanations ?? the mail refer to LS cylinder and to LS based MZC**. Noticed that these methods give only an upper bound on the cylindricity while our method give a precise evaluation with a bound on the error.

For our technique, we give at each iteration:
— the evaluated cylindricity,
— the upper bound on the error on the cylindricity obtained by Theorem 5,
— the upper bound θ on the parameters on the cylinder, this bound is given

at first iteration and computed using Theorem 5 after,
— $\max(\sqrt{\alpha^2+\beta^2}, \sqrt{u^2+v^2}, |\eta|, |\tau|)$ denoted as $|\mathbf{p}|$ and used to compute θ at the next iteration.

This technique has been tried on real data points obtained by actual measurement. **????? Details on the objects measured ????????** For the first object, the six points determining the minimum zone hyperboloid remains identical during all the iteration of our method, thus the iterative process is just improving the curvature of the hyperboloid so that it is closer to the cylinder. The second object of very good quality is more interesting. Since the points are more cocylindrical, the points defining the minimum zone hyperboloid change with the iterations.

Well, I have in fact a problem with the results given in the mail. Third iteration claimes that 19.02 ¡ cylindricity ¡ 19.04 and fourth iteration gives 19.46 ¡ cylindricity ¡ 19.48

Method		R	h	cylindricity	error	θ	$\lvert\mathbf{p}\rvert$
		(inches)		(μ inches)			
LS				$\leq$3035			
LS based				$\leq$2463			
this paper	1	2.003	2	2069	200	10000	4330
$\sharp$ iterations	2			2066	127	7956	1634
	3			2066	50	5003	1033
	4			2066	34	4134	1033
LS				$\leq$37.2			
LS based				$\leq$26.8			
this paper	1	0.625	1.25	19.6	6.4	1000	400
$\sharp$ iterations	2			18.8	1.3	450	9.7
	3			19.03	0.01	42	9.6
	4			19.47	0.01	37	10
	5			19.01	0.01	37	9.9
	6			19.48	0.01	37	10

7 Conclusion

We have shown that in two dimensions the minimum area annulus gives an approximation of the minimal width with an error $O(\omega^2/R)$ and an analogous result has been established in three dimensions for the cylindricity problem. If ω is not small enough (i.e., the physical object is of poor quality, so that its zone cylinder is "thick"), the linear programming approach does not produce a good approximation of the desired solution.

In two dimensions, Ramos [5] proves that the exact minimum width annulus can always be determined by solving an LP-type problem. This would suggest that, to avoid the described limitation of our approach, one may try to establish that finding the minimum zone hyperboloid in the family

$$(r+\tau)^2 \leq x^2 + y^2 - 2uxz - 2vyz - 2\alpha x - 2\beta y \leq (R+\eta)^2$$

is an LP-type problem (where R and r are fixed), so that iteration may be possible without assuming a small width.

Finally, we remark that our scheme applies also to the construction of the minimum enclosing cylinder or the maximal inscribed cylinder, by minimizing η or $-\tau$ instead of $\eta - \tau$ in the linear program. In such cases, however, the quantity being optimized is the area of an horizontal cross-section and not the radius. Nevertheless, under reasonable hypotheses, we expect that in these cases as well provably good approximations can be obtained.

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References

[1] Pankaj K. Agarwal, Boris Aronov, Sariel Har-Peled, and Micha Sharir. Approximation and exact algorithms for minimum-width annuli and shells. In *Proc. 15th Annu. ACM Sympos. Comput. Geom.*, pages 380–389, 1999.

[2] Pankaj K. Agarwal, Boris Aronov, and Micha Sharir. Line traversals of balls and smallest enclosing cylinders in three dimensions. In *Proc. 8th ACM-SIAM Sympos. Discrete Algorithms*, pages 483–492, 1997.

[3] Pankaj K. Agarwal and Micha Sharir. Efficient randomized algorithms for some geometric optimization problems. *Discrete Comput. Geom.*, 16:317–337, 1996.

[4] Jesús García-López, Pedro Ramos, and Jack Snoeyink. Fitting a set of points by a circle. *Discrete Comput. Geom.*, 20:389–402, 1998.

[5] Pedro A. Ramos. Computing roundness is easy if the set is almost round. In *Proc. 15th Annu. ACM Sympos. Comput. Geom.*, pages 307–315, 1999.

[6] T. J. Rivlin. Approximating by circles. *Computing*, 21:93–104, 1979.

[7] Micha Sharir and Emo Welzl. A combinatorial bound for linear programming and related problems. In *Proc. 9th Sympos. Theoret. Aspects Comput. Sci.*, volume 577 of *Lecture Notes Comput. Sci.*, pages 569–579. Springer-Verlag, 1992.